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"ANALYSIS OF THE LINE GRAPH OF STAR GRAPH AND WHEEL GRAPH".

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Abstract:
This paper looks at the features of line graphs made from star graphs and wheel graphs. We carefully compare how these line graphs are similar and different. We figure out important details like how the nodes are connected, how many connections each node and the graph so no connected nodes. We also talk about how these findings can be useful in areas like graph theory.

Keywords: Line graph, Star graph, Wheel graph.

1. Introduction:

Graph theory is a critical area of mathematics with numerous applications in computer science, network analysis, and combinatorial optimization. In this paper, we focus on two specific types of graphs: star graphs and wheel graphs, and their respective line graphs. The star graph, denoted as S_n consists of one central node connected to n peripheral nodes. The wheel graph, denoted as W_n, consists of a cycle of n nodes with an additional central node connected to all nodes in the cycle. The line graph $L(G)$ of a graph G is a graph such that each vertex of $L(G)$ represents an edge of G, and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in$G$.
Graphs can be classified in many ways, such as directed or undirected, weighted or unweighted, and simple or multigraphs.

Directed Graphs: Edges have a direction, indicating a one-way relationship.
Undirected Graphs: Edges do not have a direction, indicating a two-way relationship.
Weighted Graphs: Edges have weights, representing the cost or distance between vertices.
Unweighted Graphs: Edges do not have weights.

Definition of a Graph

In mathematics and computer science, a graph is a structure used to model pairwise relationships between objects. It consists of two main components:

1. **Vertices (or Nodes)**: These are the fundamental units or points of the graph. Each vertex represents an entity or object.
2. **Edges (or Links)**: These are the connections between pairs of vertices. Each edge represents a relationship or interaction between the entities represented by the vertices.

Formally, a graph G is defined as a pair $G = (V, E)$, where:

- V is a set of vertices.
- E is a set of edges, each of which is a pair of vertices from V

Line Graph

A line graph, also known as an edge graph, is a graph that represents the adjacencies between edges of another graph. The line graph $L(G)$ of a graph G is constructed as follows:

- Each vertex of $L(G)$ represents an edge of G.
- Two vertices in $L(G)$ are adjacent if and only if their corresponding edges in G share a common vertex.

Visual Representations

1. Graph: Generally represented with vertices as dots and edges as lines connecting the dots.

2. Line Graph: Derived from the original graph by representing each edge as a vertex and connecting vertices that share a common endpoint in the original graph.

3. Star Graph: Imagined with a central node connected to several outer nodes, forming a star-like shape.

4. Wheel Graph: Looks like a wheel with a central hub connected to a cycle of nodes around it.

2. Review of literature:

Frank Harary's (4) is one of the foundational texts in graph theory. It analytically covers various topics, including subgraphs, connectivity and coloring with hard proofs and many examples. The clarity and thoroughness of this have made it a must-read for anyone interested in graph theory. The next (5) It covers topics like matching, network flows and planar graphs in difficulty, making it suitable for both beginners and advanced. After that thorough introduction to graph theory. It covers standard topics with a variety of examples and exercises that are useful for scholars.

(8) on the practical applications of graph theory in various fields. It includes many real-world problems and shows how graph theory can be used to solve them. Ranganathan's (10) is another excellent introduction to graph theory. It covers a broad range of topics and includes many examples and exercises to illustrate the concepts. Its clear presentation and comprehensive coverage make it a useful resource for scholars. We will observe and draw the star graph and wheel graph convert to line graph and find the results.

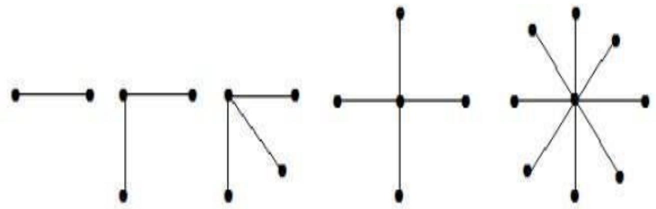
3. Line Graph of the Star Graph

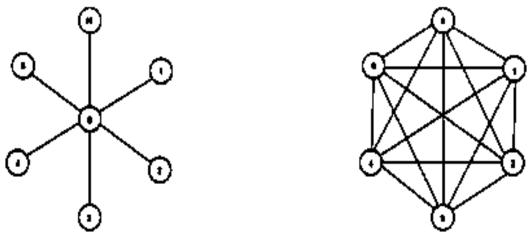
3.1 Definition and Properties

A star graph S_n is a complete bipartite graph $K_{1,n}$ consisting of one central node (called the hub) connected to n outer nodes (called leaves), but with no edges between the leaves. Star graphs have the following characteristics:

- They have $n + 1$ vertices and n edges.
- The degree of the central vertex (hub) is n, while the degree of each leaf is 1.

The star graph S_n has one central vertex V_o and n peripheral vertices $V_1, V_2, \ldots . V_n$, with edges $e_i = (V_{0\,,} V_i)$ for $i = 1, 2, \ldots ., n$.





Line Graph of $S3$ (Star graph with 3 leaves)

Vertices of $S3$**:**$\{v0, v1, v2, v3\}$

Edges of $S3$**:**

$$\{v0, v1\}, \{v0, v2\}, \{v0, v2, \{v0, v3\}$$

Vertices of $L(S3)$: 3 vertices, each representing an edge of $S3$: $\{e1, e2, e3\}$

Edges of $L(S3)$: Each vertex is connected to every other vertex, forming a complete graph $K3$.

Line Graph of $S4$(Star graph with 4 leaves)

Vertices of $S4$: $\{v0, v1, v2, v3, v4\}$

Edges of $S4$: $\{v0, v1\}, \{v0, v2\}, \{v0, v3\}, \{v0, v4\}$

Vertices of $L(S4)$: 4 vertices, each representing an edge of $S4$:$\{e1, e2, e3, e4\}$

Edges of $L(S4)$: Each vertex is connected to every other vertex, forming a complete graph $K4$.

Line Graph of $S5$(Star graph with 5 leaves)

Vertices of $S5$: $\{v0, v1, v2, v3, v4, v5\}$

Edges of $S5$: $\{v0, v1\}, \{v0, v2\}, \{v0, v3\}, \{v0, v4\}, \{v0, v5\}$

Vertices of $L(S5)$: 5 vertices, each representing an edge of $S5$: $\{e1, e2, e3, e4, e5\}$

Edges of $L(S5)$: Each vertex is connected to every other vertex, forming a complete graph $K5$.

Line Graph of S6 (Star graph with 6 leaves)

Vertices of $S6$: $\{v0, v1, v2, v3, v4, v5, v6\}$

Edges of $S6$: $\{v0, v1\}, \{v0, v2\}, \{v0, v3\}, \{v0, v4\}, \{v0, v5\}, \{v0, v6\}$

Vertices of $L(S6)$: 6 vertices, each representing an edge of $S6$: $\{e1, e2, e3, e4, e5, e6\}$

Edges of L(S6)**:**Each vertex is connected to every other vertex, forming a complete graph $K6$.

Summary for Line Graphs of Star Graphs

Line Graph of $S3$: $L(S3)$is $K3$ with 3 vertices and 3 edges.
Line Graph of $S4$: $L(S4)$is$K4$ with 4 vertices and 6 edges.
Line Graph of $S5$: $L(S5)$is$K5$with 5 vertices and 10 edges.
Line Graph of $S6$: $L(S6)$ is $K6$with 6 vertices and 15 edges.
In each case, the line graph of a star graph is a complete graph where each vertex is connected to every other vertex.

3.2 Construction of the Line Graph

The line graph $L(S_n)$ has n vertices, each corresponding to one of the n edges of S_n. Since each edge of S_n shares the central vertex $V_{0\,,}$, the line graph $L(S_n)$ is a complete graph K_n.

3.3 Adjacency Matrix

The adjacency matrix A of $L(S_n) = K_n$is given by:

$$A_{ij} = \begin{cases} 1 \text{ if } i \neq j \\ 0 \text{ if } i = j \end{cases}$$

3.4 Degree Sequence

The degree sequence of $L(S_n)$ is $n-1,\ n-1, \dots\dots. n-1$.

3.5 Chromatic Number

The chromatic number of $L(S_n) = K_n$ is n.

4. Line Graph of the Wheel Graph

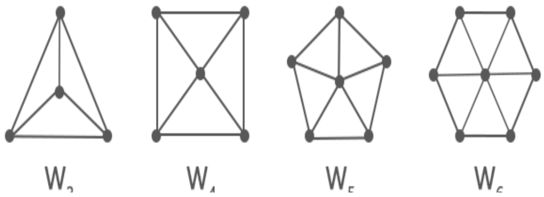
Wheel Graph

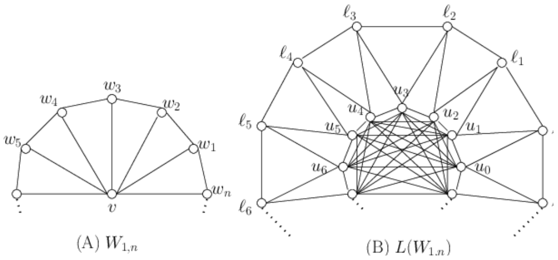
A wheel graph $W_n$is a graph formed by connecting a single central vertex to all vertices of a cycle graph C_{n-1}. The wheel graph W_n has the following properties:

- It has n vertices and $2(n-1)$ edges.
- The degree of the central vertex is $n-1$.
- The degree of each vertex on the cycle is 3.

4.1 Definition and Properties

The wheel graph W_n consists of a cycle C_n with n vertices and an additional central vertex $V_{0\,,}$ connected to all vertices in C_n.





(A) $W_{1,n}$ (B) $L(W_{1,n})$

Examples:

Line Graph of W3 (Wheel graph with 3 vertices)

Vertices of $W3$**:**$\{v0, v1, v2\}$

Edges of $W3$:

Cycle edges: $\{v1, v2\}, \{v2, v3\}, \{v3, v1\}$

Rods: $\{v0, v1\}, \{v0, v2\}, \{v0, v3\}$

Vertices of $L(W3)$**:** 4 vertices, each representing an edge of $W3$:$\{e1, e2, e3, e4\}$ where e_i represents one of the 4 edges.

Edges of $L(W3)$**:**Each vertex is connected to every other vertex, forming a complete $K4$.

Line Graph of $W4$ **(Wheel graph with 4 vertices)**

Vertices of $W4$:$\{v0, v1, v2, v3\}$

Edges of $W4$:

Cycle edges: $\{v1, v2\}, \{v2, v3\}, \{v3, v4\}, \{v4, v1\}$

Rods: $v0, v1, v0, v2, v0, v3, v0, v4$

Vertices of $L(W4)$: 6 vertices, each representing an edge of $W4$:$\{e1, e2, e3, e4, e5, e6\}$ where e_i represents one of the 6 edges.

Edges of $L(W4)$:

Each cycle edge vertex is adjacent to 4 other vertices.

Each spoke edge vertex is adjacent to 2 cycle edge vertices and 2 other rod edge vertices.

Line Graph of $W5$ (Wheel graph with 5 vertices)

Vertices of $W5$: $\{v0, v1, v2, v3, v4\}$

Edges of $W5$:

Cycle edges: $\{v1, v2\}, \{v2, v3\}, \{v3, v4\}, \{v4, v5\}, \{\mathrm{v}5, \mathrm{v}1\}$

Rods: $\{v0, v1\}, \{v0, v2\}, \{v0, v3\}, \{v0, v4\}, \{v0, v5\}$

Vertices of L(W5)**:** 8 vertices, each representing an edge of $W5$:$\{e1, e2, e3, e4, e5, e6, e7, e8\}$where e_i represents one of the 8 edges.

Edges of $L(W5)$:

Each cycle edge vertex is adjacent to 4 other vertices.

Each spoke edge vertex is adjacent to 2 cycle edge vertices and 3 other rod edge vertices.

Line Graph of W6(Wheel graph with 6 vertices)

Vertices of $W6$: $\{v0, v1, v2, v3, v4, v5\}$

Edges of $W6$**:**

Cycle edges: $\{v1, v2\}, \{v2, v3\}, \{v3, v4\}, \{v4, v5\}, \{v5, v6\}, \{v6, v1\}$

Rods: $\{v0, v1\}, \{v0, v2\}, \{v0, v3\}, \{v0, v4\}, \{v0, v5\}, \{v0, v6\}$

Vertices of $L(W6)$: 10 vertices, each representing an edge of $W6$:$\{e1, e2, e3, e4, e5, e6, e7, e8, e9, e10\}$ where e_i represents one of the 10 edges.

Edges of $L(W6)$:

Each cycle edge vertex is adjacent to 4 other vertices.

Each spoke edge vertex is adjacent to 2 cycle edge vertices and 4 other rod edge vertices.

Line Graphs of Wheel Graphs
Line Graph of W3: 4 vertices, forming a complete graph K4.
Line Graph of W4: 6 vertices, forming a very interrelated graph.
Line Graph of W5: 8 vertices, forming a very interrelated graph.
Line Graph of W6: 10 vertices, forming a very interrelated graph.

4.2 Construction of the Line Graph

The line graph $L(W_n)$ is constructed by considering each edge in W_n. There are n edges connecting the central vertex to the cycle vertices and n edges in the cycle. Thus, $L(W_n)$ has $2n$ vertices.

4.3 Adjacency Matrix

The adjacency matrix A of $L(W_n)$ is more complex due to the combination of star and cycle structures. Let $e_i = (V_{0\,,} V_i)$ for $i = 1, 2, \ldots., n$ for and $(\ V_i,\ V_{i+1})$ for $i = 1, 2, \ldots.. n-1$, $f_n = (V_{n,} V_1)$. The adjacency relations are:

$$A_{ij} = \begin{cases} 1 \text{ if edges } e_i \text{ and } e_j \text{ share } V_{0\,,} \\ 1 \text{if edges } f_i \text{ and } f_j \text{ share a vertex in } C_{n\,,} \\ 1 \text{ if edges } e_i \text{ and } f_i \text{ share } V_{i\,,} \\ 0 \text{ otherwise} \end{cases}$$

for $i, j = 1, 2, \ldots\ldots, 2n.$

4.4 Degree Sequence

The degree sequence of $L(W_n)$ varies. For vertices corresponding to the edges in the cycle, the degree is 3. For vertices corresponding to the edges connected to the central vertex, the degree is 2.

4.5 Chromatic Number

The chromatic number of $L(W_n)$ depends on n. It requires careful analysis but generally relates to the chromatic properties of cycles and stars.

5. Comparative Analysis

5.1 Structural Differences

The mainoperational difference between $L(S_n)$and $L(W_n)$ lies in their adjacency relations. $L(S_n)$ forms a complete graph, while $L(W_n)$ has a more involved structure due to the relationship between the cycle and the fundamental node.

5.2 Applications

Understanding the line graphs of star and wheel graphs has applications in graph design, particularly in the efficient arrangement and statement of star and wheel graphanalysis. Also, these understandings are useful in solving problems related to graph coloring and network reliability.

Theorem 1: The Line Graph of a Star Graph S_n is a Complete Graph K_n

Proof:

Let S_n be a star graph with a central vertex $V_{0\,,}$ and n peripheral vertices $V_1,\ V_2, \dots . V_n$. The edges of S_n are $e_i = (V_{0\,,}V_i)$for $i\ =\ 1, 2, \dots . , n$.
The line graph $L(S_n)$ is constructed as follows:
1. Each vertex in $L(S_n)$ corresponds to an edge in S_n.
2. Two vertices in $L(S_n)$ are adjacent if and only if their corresponding edges in S_n share a common vertex.

In S_n, every edge shares the central vertex $V_{0\,,}$. Therefore, every pair of edges $(e_{i\,,}e_j\ )$ for $i \neq j$ shares the vertex $V_{0\,,}$. Hence, in $L(S_n)$, every pair of vertices $(V_{i\,,}V_j\ )$ for $i \neq j$ is adjacent.

Thus, $L(S_n)$, is a complete graph with n vertices, which is denoted as K_n.

Theorem 2: The line graph $L(S_n)$, of a star graph S_n has $\binom{n}{2}$ edges.

Proof:

1. Star Graph Edges:The star graph $S_n$has n edges, each connecting the central vertex v to one of the$n$ outer vertices.

2. Vertices of Line Graph $L(S_n)$:The vertices $L(S_n)$, correspond to the $n$edges of S_n.

3. Edges of Line Graph $L(S_n)$:In $L(S_n)$, each pair of verticese_i and e_j is connected by an edge if their corresponding edges in S_n share a common vertex.
Since all $n$edges in $S_n$share the central vertex$v$, every pair of vertices in$L(S_n)$is connected by an edge.

4. Number of Edges in Complete Graph K_n:A complete graph K_n has$\binom{n}{2}$ edges, where$\binom{n}{2} = \frac{n(n-1)}{2}$.
This is the number of ways to choose 2 vertices out of $n$vertices to form an edge.

We have to prove Since$L(S_n)$, is a complete graph K_n it has$\binom{n}{2}$edges.

Theorem 3: The Line Graph of a Wheel Graph W_nContains a $3n - 3$ - Cycle

Proof:

Let W_n be a wheel graph with a central vertex V_0, a cycle C_n consisting of vertices $V_1,\ V_2, \dots . V_n$ and edges $(V_{0\,,}V_i)$ for $i\ =\ 1, 2, \dots, n$, along with the edges forming the cycle C_n.

The line graph $L(W_n)$ is constructed as follows:

1. Each vertex in $L(W_n)$ corresponds to an edge in W_n.
2. Two vertices in $L(W_n)$ are adjacent if and only if their corresponding edges in W_n share a common vertex.

Consider the edges in W_n in two parts:
1. Edges $e_i = (V_0, V_i)$ for $i = 1, 2, \ldots, n$,.
2. Edges $f_i = (V_i, V_{i+1})$ for $i = 1, 2, \ldots .. n-1$, $f_n = (V_n, V_1)$ in $L(W_n)$:
1. Each vertex corresponding to e_i is adjacent to every vertex corresponding to f_i and f_{i-1} with *(*$f_0 = f_n$*)*
2. Each vertex corresponding to f_i is adjacent to f_{i-1} and f_{i+1}, forming a cycle of length n.

Thus, in $L(W_n)$, there are:
n vertices corresponding to e_i.
n vertices corresponding to f_i.

Each of these vertices is connected in such a way that there are cycles formed by the adjacency rules, resulting in multiple cycles of length related to n. Specifically, the total cycle length would be $3n - 3$ due to the nature of connections between e_i and f_i.

Therefore, $L(W_n)$, contains a cycle of length $3n - 3$.

Theorem 4: The line graph $L(W_n)$, of a wheel graph W_n contains a cycle of length $2(n - 1)$.

Proof:

1. Wheel Graph Definition:As stated earlier, W_n consists of a cycleC_{n-1} and an additional central vertex v connected to each vertex of the cycle.
The vertex set $V(W_n) = \{v, u_1, u_2, \ldots\ldots, u_{n-1}\}$.
The edge set $E(W_n) = \{(v, u_i) \mid 1 \leq i \leq n-1\} \cup \{(u_i, u_{i+1}) \mid 1 \leq i \leq n-2\} \cup \{(u_{n-1}, u_1)\}$.

2. Vertices of Line Graph $L(W_n)$: The vertices of $L(W_n)$correspond to the edges of (W_n).

3. Edges of Line Graph $L(W_n)$:Two vertices in $L(W_n)$,:remain adjacent if and only if their corresponding edges in W_n share a common vertex.

4. Identifying the Cycle:Consider the sequence of edges$(v, u1), (u1, u2), (v, u2), (u2, u3), \ldots, (v, u_{n-1}), (u_{n-1}, u1)$ in W_n.
These edges alternate between edges connecting$v$ to cycle vertices and edges between consecutive cycle vertices.
This sequence forms a cycle in $L(W_n)$,:because each edge shares a common vertex with the next edge in the sequence.

5. Cycle Length:
The sequence includes$(n-1)$ edges connecting $v$to cycle vertices and $(n-1)$edges connecting consecutive cycle vertices.
Hence, the total length of the cycle in $L(W_n)$,: is $2(n-1)$.

Therefore, $L(W_n)$:contains a cycle of length $2(n-1)$.

6. Conclusion

This paper has on condition that a thorough examination of the line graphs of star and wheel graphs. We resulting their adjacency matrices, degree sequences and chromatic numbers, importance the differences and similarities. The findings have important implications in graph theory and other applied fields.

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